

# Noether-Lefschetz locus and generalisation of an example due to Mumford

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## Abstract

In this article we generalize the well-known example due to Mumford for a generically non-reduced component of the Hilbert scheme of curves in  $\mathbb{P}^3$  whose general element is smooth. The example considers smooth curves in smooth cubic surfaces in  $\mathbb{P}^3$ . In this article we give similar examples of generically non-reduced component of the Hilbert scheme of curves in  $\mathbb{P}^3$ , for every integer  $d \geq 5$ , whose general element is a smooth curve contained in a smooth degree  $d$  surface in  $\mathbb{P}^3$  and not in any surface of smaller degree. The techniques used are motivated by the study of Noether-Lefschetz locus.

## 1 Introduction

With Grothendieck's construction of the Hilbert scheme one can give a scheme structure to families of curves, which up to then were described only as algebraic varieties. In 1962, only a few years after Grothendieck introduced the Hilbert scheme, Mumford [Mum62] showed that there exists generically non-reduced (in the sense, the localization of the structure sheaf at every point contains a non-trivial nilpotent element) irreducible components of the Hilbert scheme of curves in  $\mathbb{P}^3$  such that a general element is a smooth curve contained in a cubic surface in  $\mathbb{P}^3$ . This example inspired the investigation of such components. Kleppe shows in [Kle81] that an irreducible component  $L$  of the Hilbert scheme of curves parametrizing smooth curves contained in a cubic surface in  $\mathbb{P}^3$  is non-reduced if and only if for a general  $C \in L$  and a smooth cubic surface  $X$  containing  $C$ ,  $h^1(\mathcal{O}_X(-C)(3)) \neq 0$ . Using this condition he gives examples in [Kle85] of such non-reduced components. In this article we generalize these results. We give examples

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for each integer  $d \geq 5$ , non-reduced components of the Hilbert scheme of smooth space curves contained in a smooth degree  $d$  surface in  $\mathbb{P}^3$  but not in any surface (in  $\mathbb{P}^3$ ) of smaller degree. The main tool used in this article comes from the study of Hodge loci, which contrasts previous approaches.

We recall briefly the main ideas used in [Kle81] to illustrate the difficulty in producing such examples. The most important observation is that for a smooth curve  $C$  in a smooth cubic surface  $X$  in  $\mathbb{P}^3$   $H^1(\mathcal{N}_{C|X}) = 0$  (since  $H^1(\mathcal{N}_{C|X})^\vee \cong H^0(\mathcal{N}_{C|X}^\vee \otimes K_C)$  where  $K_C$  is the canonical divisor and  $\deg(\mathcal{N}_{C|X}^\vee \otimes K_C) = 2\rho_a(C) - 2 - C^2 = -\deg(C) < 0$ ). Therefore, the natural morphism from  $H^0(\mathcal{N}_{C|\mathbb{P}^3})$  to  $H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C)$  is surjective (use the normal short exact sequence). This means (using the natural pull-back morphism, say  $\rho'$  from  $H^0(\mathcal{N}_{X|\mathbb{P}^3})$  to  $H^0(\mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C)$  and basic knowledge of flag Hilbert scheme) for any infinitesimal deformation of  $X$  in  $\mathbb{P}^3$ , there exists a corresponding infinitesimal deformation of  $C$  contained in this. Furthermore, if  $\rho'$  is not surjective then there exists an infinitesimal deformation of  $C$  not corresponding to any infinitesimal deformation of  $X$ . This condition is equivalent to  $h^1(\mathcal{O}_X(-C)(3)) \neq 0$  (use  $\mathcal{N}_{X|\mathbb{P}^3} \cong \mathcal{O}_X(3)$  and  $H^1(\mathcal{O}_X(3)) = 0$ ). An easy dimension count tells us that this is a necessary and sufficient condition for the corresponding irreducible component to be non-reduced (an important assumption used in this step is that a curve corresponding to a general point in this component is contained in a cubic surface in  $\mathbb{P}^3$ ). For  $d \geq 5$ ,  $H^1(\mathcal{N}_{C|X})$  for a smooth curve  $C$  in  $X$  is never zero. So, it is not possible to duplicate this approach for  $d \geq 5$ , i.e., for finding non-reduced irreducible components of the Hilbert scheme of smooth curves contained in a smooth degree  $d$  surface not contained in a surface of degree smaller than  $d$ .

We instead use results from the theory of Noether-Lefschetz locus to produce such examples. There are numerous examples of non-reduced components of the Noether-Lefschetz locus (see [Dan14, Theorems 6.16, 6.17]) which is the starting point for our study. Moreover, the tangent space at a point on the Noether-Lefschetz locus has an explicit description in terms of commutative algebra (see [Voi03, §6.2]). This suggest that using standard computer programming one can produce further examples of non-reduced components of the Noether-Lefschetz locus which would in turn give new examples of non-reduced irreducible components of the Hilbert scheme of smooth space curves. However, the second approach has not been explored in this article.

The first main result in this article gives a cohomological criterion for the existence of the

aforementioned components. But before we proceed, we need to recall the notion of Hodge locus. Consider,  $U_d \subseteq \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)))$  the open subscheme parametrizing smooth projective hypersurfaces in  $\mathbb{P}^3$  of degree  $d$ . Let  $\mathcal{X} \xrightarrow{\pi} U_d$  be the corresponding universal family. For a given  $F \in U_d$ , denote by  $X_F$  the surface  $X_F := \pi^{-1}(F)$ . Let  $X \in U_d$  and  $U \subseteq U_d$  be a simply connected neighbourhood of  $X$  in  $U_d$  (under the analytic topology). Then  $\pi|_{\pi^{-1}(U)}$  induces a variation of Hodge structure  $(\mathcal{H}, \nabla)$  on  $U$  where  $\mathcal{H} := R^2\pi_*\mathbb{Z} \otimes \mathcal{O}_U$  and  $\nabla$  is the Gauss-Manin connection. Note that  $\mathcal{H}$  defines a local system on  $U$  whose fiber over a point  $F \in U$  is  $H^2(X_F, \mathbb{Z})$  where  $X_F = \pi^{-1}(F)$ . Consider a non-zero element  $\gamma_0 \in H^2(X_F, \mathbb{Z}) \cap H^{1,1}(X_F, \mathbb{C})$  such that  $\gamma_0 \neq c_1(\mathcal{O}_{X_F}(k))$  for  $k \in \mathbb{Z}_{>0}$ . This defines a section  $\gamma \in (\mathcal{H} \otimes \mathbb{C})(U)$ . Let  $\overline{\gamma}$  be the image of  $\gamma$  in  $\mathcal{H}/F^2(\mathcal{H} \otimes \mathbb{C})$ . The Hodge loci, denoted  $\text{NL}(\gamma)$  is then defined as

$$\text{NL}(\gamma) := \{G \in U | \overline{\gamma}_G = 0\},$$

where  $\overline{\gamma}_G$  denotes the value at  $G$  of the section  $\overline{\gamma}$ . See [Voi03, §5] for a detailed study of the subject. We then prove,

**Theorem 1.1.** Let  $X$  be a smooth degree  $d$  surface,  $\gamma \in H^{1,1}(X, \mathbb{Z})$  and  $C$  be a smooth semi-regular curve in  $X$  such that  $\gamma - [C]$  is a multiple of the class of the hyperplane section  $H_X$ , where  $[C]$  is the cohomology class of  $C$ . Let  $P'$  be the Hilbert polynomial of  $C$ . If  $h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))$  and  $\overline{\text{NL}(\gamma)}$  (closure taken in  $U_d$  under Zariski topology) is irreducible generically non-reduced then there is an irreducible generically non-reduced component of the Hilbert scheme  $\text{Hilb}_{P'}$  containing  $C$  and parametrizing smooth curves in  $\mathbb{P}^3$ . Furthermore, given  $\gamma$  there *always* exists such a  $C$  (i.e., smooth, semi-regular and  $\gamma - [C]$  is a multiple of  $H_X$ ) and  $C$  is *not* contained in a surface of degree less than  $d$ .

See Theorems 3.3 and 4.1 for further details.

Combining this result with a result from Noether-Lefschetz locus proven in [Dan14] we conclude that

**Theorem 1.2.** For  $d \geq 5$  and  $m \gg d$ , there exists a generically non-reduced irreducible component of the Hilbert scheme parametrizing smooth curves in  $\mathbb{P}^3$ :

1. of degree  $md + 3$  and the arithmetic genus  $1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)$ ,

2. a generic element in this component corresponds to a smooth curve contained in a smooth degree  $d$  surface in  $\mathbb{P}^3$  but not in any surface of smaller degree

See Corollary 4.3 and Lemma 4.4 for a proof of the statement.

One of the main ideas that we exploit is that the Hodge locus  $\text{NL}(\gamma)$  of the cohomology class, say  $\gamma$  of a divisor  $D$  on a surface  $X$  is invariant if we translate it by a multiple of the hyperplane section. By twisting the line bundle  $\mathcal{O}_X(D)$  by some multiple of the hyperplane section, we can conclude that a general curve in the resulting linear system is smooth and semi-regular in the sense of Bloch (see [Blo72]). Then the Hodge locus and the flag Hilbert schemes are closely related. In particular, if we denote by  $P$  the Hilbert polynomial of this curve, there exists an irreducible component  $H_\gamma$  of the flag Hilbert scheme  $\text{Hilb}_{P,Q_d}$  such that  $\text{pr}_2(H_\gamma)_{\text{red}} \cong \overline{\text{NL}(\gamma)}_{\text{red}}$  and  $H_\gamma$  is non-reduced if and only if so is  $\overline{\text{NL}(\gamma)}$ . The only point that needs to be checked is that  $\text{pr}_1(H_\gamma)$  is in fact an irreducible component of the Hilbert scheme of curves corresponding to  $P$ . This is shown in Proposition 3.2.

**Notation 1.3.** We fix once and for all a few notations that will be used throughout this article. By a *surface* or a *curve* we mean a scheme of pure dimension 2 or 1, respectively in  $\mathbb{P}^3$ . For a Hilbert polynomial  $P$  of a curves or a surface in  $\mathbb{P}^3$ , denote by  $H_P$  the Hilbert scheme parametrizing all subschemes in  $\mathbb{P}^3$  with Hilbert polynomial  $P$ . Denote by  $Q_d$  the Hilbert polynomial of a degree  $d$  surface in  $\mathbb{P}^3$ . For a pair of Hilbert polynomials  $P, Q_d$ , denote by  $H_{P,Q_d}$  the corresponding flag Hilbert scheme.

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## 2 Preliminaries

**2.1.** In this section we recall the basic definitions of Noether-Lefschetz locus. See [Voi02, §9, 10] and [Voi03, §5, 6] for a detailed presentation of the subject.

**Definition 2.2.** Recall, for a fixed integer  $d \geq 5$ , the *Noether-Lefschetz locus*, denoted  $\text{NL}_d$ , parametrizes the space of smooth degree  $d$  surfaces in  $\mathbb{P}^3$  with Picard number greater than 1.

Using the Lefschetz  $(1, 1)$ -theorem this is the parametrizing space of smooth degree  $d$  surfaces with  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}) \neq \mathbb{Z}$ .

**Notation 2.3.** Let  $X \in U_d$  and  $\mathcal{O}_X(1)$ , the very ample line bundle on  $X$  determined by the closed immersion  $X \hookrightarrow \mathbb{P}^3$  arising (as in [Har77, II.Ex.2.14(b)]) from the graded homomorphism of graded rings  $S \rightarrow S/(F_X)$ , where  $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^3})$  and  $F_X$  is the defining equations of  $X$ . Denote by  $H_X$  the very ample line bundle  $\mathcal{O}_X(1)$ . Denote by  $H^2(X, \mathbb{C})_{\text{prim}}$  the primitive cohomology. Given  $\gamma \in H^2(X, \mathbb{C})$ , denote by  $\gamma_{\text{prim}}$  the image of  $\gamma$  under the natural morphism from  $H^2(X, \mathbb{C})$  to  $H^2(X, \mathbb{C})_{\text{prim}}$ . Since the very ample line bundle  $H_X$  remains of type  $(1, 1)$  in the family  $\mathcal{X}$ , we can therefore conclude that  $\gamma \in H^{1,1}(X)$  remains of type  $(1, 1)$  if and only if  $\gamma_{\text{prim}}$  remains of type  $(1, 1)$ . In particular,  $\text{NL}(\gamma) = \text{NL}(\gamma_{\text{prim}})$ .

**2.4.** Note that,  $\text{NL}_d$  is a countable union of subvarieties. Every irreducible component of  $\text{NL}_d$  is locally of the form  $\text{NL}(\gamma)$  for some  $\gamma \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ ,  $X \in \text{NL}_d$  such that  $\gamma_{\text{prim}} \neq 0$ . There is a natural analytic scheme structure on  $\overline{\text{NL}(\gamma)}$  (see [Voi03, Lemma 5.13]).

**Definition 2.5.** We now discuss the tangent space to the Hodge locus,  $\text{NL}(\gamma)$ . We know that the tangent space to  $U$  at  $X$ ,  $T_X U$  is isomorphic to  $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ . This is because  $U$  is an open subscheme of the Hilbert scheme  $H_{Q_d}$ , the tangent space of which at the point  $X$  is simply  $H^0(\mathcal{N}_{X|\mathbb{P}^3})$ . Given the variation of Hodge structure above, we have (by Griffith's transversality) the differential map:

$$\overline{\nabla} : H^{1,1}(X) \rightarrow \text{Hom}(T_X U, H^2(X, \mathcal{O}_X))$$

induced by the Gauss-Manin connection. Given  $\gamma \in H^{1,1}(X)$  this induces a morphism, denoted  $\overline{\nabla}(\gamma)$  from  $T_X U$  to  $H^2(\mathcal{O}_X)$ .

**Lemma 2.6** ([Voi03, Lemma 5.16]). The tangent space at  $X$  to  $\text{NL}(\gamma)$  is equal to  $\ker(\overline{\nabla}(\gamma))$ .

Another important notion that will be used in this article is that of semi-regularity. We recall first the definition.

**Definition 2.7.** Let  $X$  be a surface and  $C \subset X$ , a curve in  $X$ . Since  $X$  is smooth,  $C$  is local complete intersection in  $X$ . Denote by  $i$  the closed immersion of  $C$  into  $X$ . This gives rise to the short exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{N}_{C|X} \rightarrow 0 \tag{1}$$

where  $\mathcal{N}_{C|X}$  is the normal sheaf of  $C$  in  $X$ . The *semi-regularity map*  $\pi$  is the boundary map from  $H^1(\mathcal{N}_{C|X})$  to  $H^2(\mathcal{O}_X)$ . We say that  $C$  is *semi-regular* if  $\pi$  is injective.

**Theorem 2.8** ([Dan14, Theorem 4.8]). Let  $X$  be a smooth degree  $d$  surface and  $C$  be a curve in  $X$ . Let  $\gamma = [C] \in H^{1,1}(X, \mathbb{Z})$ , the cohomology class of  $C$ . Denote by  $P$  the Hilbert polynomial of  $C$ . We have the following commutative diagram

$$\begin{array}{ccccccc}
& & T_{(C,X)}H_{P,Q_d} & \longrightarrow & H^0(X, \mathcal{N}_{X|\mathbb{P}^3}) & \xrightarrow{\overline{\nabla}(\gamma)} & H^2(X, \mathcal{O}_X) \\
& & \downarrow & & \downarrow \rho_C & \circlearrowleft & \uparrow \pi_C \\
0 & \longrightarrow & H^0(C, \mathcal{N}_{C|X}) & \xrightarrow{\phi_C} & H^0(C, \mathcal{N}_{C|\mathbb{P}^3}) & \xrightarrow{\beta_C} & H^0(C, \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C) \xrightarrow{\delta_C} H^1(C, \mathcal{N}_{C|X})
\end{array}$$

where the horizontal exact sequence comes from the normal short exact sequence

$$0 \rightarrow \mathcal{N}_{C|X} \rightarrow \mathcal{N}_{C|\mathbb{P}^3} \rightarrow \mathcal{N}_{X|\mathbb{P}^3} \otimes \mathcal{O}_C \rightarrow 0,$$

$\pi_C$  is the semi-regularity map and  $\rho_C$  is the natural pull-back morphism.

**Corollary 2.9.** Let  $X$  be a smooth degree  $d$  surface in  $\mathbb{P}^3$ ,  $C \subset X$  a semi-regular curve satisfying  $H^1(\mathcal{O}_X(-C)(d)) = 0 = H^0(\mathcal{O}_X(-C)(d))$ . Then,

$$\dim T_X(\text{NL}([C])) = h^0(\mathcal{N}_{C|\mathbb{P}^3}) - h^0(\mathcal{N}_{C|X}),$$

where  $[C]$  is the cohomology class of  $C$ .

*Proof.* Notations as in the diagram in Theorem 2.8. Let  $\gamma = [C]$ . Since  $C$  is semi-regular,  $\pi_C$  is injective. It follows directly from the above theorem that,

$$T_X(\text{NL}(\gamma)) = \ker(\overline{\nabla}(\gamma)) = \ker(\delta_C \circ \rho_C) = \rho_C^{-1}(\text{Im } \beta_C).$$

Using the long exact sequence associated to

$$0 \rightarrow \mathcal{O}_X(-C)(d) \rightarrow \mathcal{O}_X(d) \rightarrow i_* \mathcal{O}_C(d) \rightarrow 0,$$

$H^1(\mathcal{O}_X(-C)(d)) = 0 = H^0(\mathcal{O}_X(-C)(d))$  implies  $\rho_C$  is an isomorphism. So,

$$\dim T_X(\text{NL}(\gamma)) = \dim \text{Im } \beta_C = h^0(\mathcal{N}_{C|\mathbb{P}^3}) - h^0(\mathcal{N}_{C|X})$$

because the kernel of the morphism  $\beta_C$  is  $H^0(\mathcal{N}_{C|X})$ .  $\square$

Recall, the following theorem which describes the relation between the Hodge locus to the cohomology class of a semi-regular curve  $C$  and deformation of a surface  $X$  containing  $C$  such that  $C$  remains a curve under deformation. a curve.

**Theorem 2.10** ([Dan14, Theorem 5.7]). Let  $X$  be a surface,  $C$  be a semi-regular curve in  $X$  and  $\gamma \in H^{1,1}(X, \mathbb{Z})$  be the class of the curve  $C$ . For any irreducible component  $L'$  of  $\overline{\text{NL}(\gamma)}$  (the closure is taken in the Zariski topology on  $U_d$ ) there exists an irreducible component  $H'$  of  $H_{P, Q_{d_{\text{red}}}}$  containing the pair  $(C, X)$  such that  $\text{pr}_2(H')$  coincides with the associated reduced scheme  $L'_{\text{red}}$ , where  $\text{pr}_2$  is the second projection map from  $H_{P, Q_d}$  to  $H_{Q_d}$ .

### 3 General criteria for non-reducedness

**3.1.** In this section we give criterion in terms of the vanishing of certain cohomology groups under which there exists irreducible, *generically non-reduced* components of the Hilbert scheme of curves in  $\mathbb{P}^3$  parametrizing *smooth* curves contained in a smooth degree  $d$  surface but not in a surface of lower degree. We later use these criteria to produce several examples.

**Proposition 3.2.** Let  $P_0$  be the Hilbert polynomial of a curve  $C$  in  $\mathbb{P}^3$ . Assume that there exists an integer  $d$  and a smooth degree  $d$  surface, say  $X$  containing  $C$ , such that  $h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))$ . Let  $L$  be an irreducible component of  $H_{P_0}$  containing  $C$ . Then, for a general element  $D \in L$ ,  $h^0(\mathcal{I}_D(d)) > 0$  i.e.,  $D$  is contained in a smooth degree  $d$  surface.

*Proof.* Denote by  $i$  the natural closed immersion of  $C$  into  $X$ . It suffices to prove that  $h^0(\mathcal{O}_C(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$ . Then by upper semi-continuity,  $h^0(\mathcal{O}_D(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$  as  $D$  varies over an open neighbourhood of  $C$  in the Hilbert scheme  $H_{P_0}$ . For  $j : D \hookrightarrow \mathbb{P}^3$ , the closed immersion, the short exact sequence

$$0 \rightarrow \mathcal{I}_D(d) \rightarrow \mathcal{O}_{\mathbb{P}^3}(d) \rightarrow j_* \mathcal{O}_D(d) \rightarrow 0$$

implies that  $h^0(\mathcal{O}_{\mathbb{P}^3}(d)) \leq h^0(\mathcal{I}_D(d)) + h^0(\mathcal{O}_D(d))$ . Hence, we have  $h^0(\mathcal{I}_D(d)) > 0$ .

Using the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-C)(d) \rightarrow \mathcal{O}_X(d) \rightarrow i_* \mathcal{O}_C(d) \rightarrow 0$$

we have,  $h^0(\mathcal{O}_X(d)) = h^0(\mathcal{O}_C(d))$  because  $h^0(\mathcal{O}_X(-C)(d)) = 0 = h^1(\mathcal{O}_X(-C)(d))$  by assumption. It then suffices to prove  $h^0(\mathcal{O}_X(d)) < h^0(\mathcal{O}_{\mathbb{P}^3}(d))$ . But this follows from the short exact sequence,

$$0 \rightarrow \mathcal{I}_X(d) \rightarrow \mathcal{O}_{\mathbb{P}^3}(d) \rightarrow \mathcal{O}_X(d) \rightarrow 0$$

and the fact that  $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$ . The proposition then follows.  $\square$

Using this result we can show the following theorem:

**Theorem 3.3.** Let  $X$  be a smooth degree  $d$  surface,  $\gamma \in H^{1,1}(X, \mathbb{Z})$  and  $C$  be a smooth semi-regular curve in  $X$ . Assume that,  $\gamma - [C]$  is a multiple of the class of the hyperplane section  $H_X$ , where  $[C]$  is the cohomology class of  $C$ . Let  $P'$  be the Hilbert polynomial of  $C$ . If  $h^1(\mathcal{O}_X(-C)(d)) = 0 = h^0(\mathcal{O}_X(-C)(d))$  and  $\overline{\text{NL}(\gamma)}$  is an irreducible generically non-reduced component of  $\text{NL}_d$  then there is an irreducible generically non-reduced component of  $H_{P'}$  containing  $C$  and parametrizing smooth curves in  $\mathbb{P}^3$ .

*Proof.* Since  $\gamma - [C]$  is a multiple of the hyperplane section  $H_X$ ,  $\overline{\text{NL}(\gamma)}$  is (scheme-theoretically) isomorphic to  $\overline{\text{NL}([C])}$ . Hence,  $\overline{\text{NL}([C])}$  is generically non-reduced.

Since  $C$  is semi-regular, Theorem 2.10 implies that there exists an irreducible component  $H_\gamma$  of  $H_{P', Q_{d_{\text{red}}}}$  such that  $\text{pr}_2(H_\gamma)$  is isomorphic to  $\text{NL}([C])_{\text{red}}$ . Denote by  $L_\gamma := \text{pr}_1(H_\gamma)$ . Notice that the fiber over  $C \in L_\gamma$  to the morphism  $\text{pr}_1 : H_\gamma \rightarrow L_\gamma$  is isomorphic to  $\mathbb{P}(I_d(C))$ . Since  $h^0(\mathcal{I}_C(d)) - 1 = h^0(\mathcal{O}_X(-C)(d)) = 0$ , we have that  $\text{pr}_1 : H_{P', Q_{d_{\text{red}}}} \rightarrow H_{P'_{\text{red}}}$  is an isomorphism onto its image on an open neighbourhood of  $C$ . Proposition 3.2 implies that there exists an open neighbourhood  $U \subset H_{P'}$  containing  $C$  which is in the image of  $\text{pr}_1$ , hence  $\text{pr}_1^{-1}(U)$  is isomorphic to  $U$ . Since  $H_\gamma$  is an irreducible component of  $H_{P', Q_{d_{\text{red}}}}$ ,  $L_\gamma$  is an irreducible component of  $H_{P'_{\text{red}}}$  and

$$\dim \overline{\text{NL}([C])} + h^0(\mathcal{O}_X(C)) - 1 = \dim H_\gamma = \dim L_\gamma,$$



for a general pair  $(C, X) \in H_\gamma$ , where the first equality follows from the fiber dimension theorem applied to the surjective projection map  $\text{pr}_2 : H_\gamma \rightarrow \overline{\text{NL}}(\gamma)$ .

Now, Corollary 2.9 implies that  $\dim T_X(\text{NL}[C]) = h^0(\mathcal{N}_{C|\mathbb{P}^3}) - h^0(\mathcal{N}_{C|X})$ . Since  $h^0(\mathcal{N}_{C|X}) = h^0(\mathcal{O}_X(C)) - 1$  (see (1)),

$$\dim T_X(\text{NL}([C])) - \dim \text{NL}([C]) = h^0(\mathcal{N}_{C|\mathbb{P}^3}) - \dim L_\gamma.$$

This implies  $h^0(\mathcal{N}_{C|\mathbb{P}^3}) > \dim L_\gamma$  for a general  $C \in L_\gamma$  because  $\text{NL}([C])$  is generically non-reduced. Since  $L_\gamma$  is an irreducible component of  $(H_{P'})_{\text{red}}$ , the corresponding component of  $H_{P'}$  is generically non-reduced. Since  $C$  is smooth, a curve corresponding to a general closed point on  $L_\gamma$  is smooth (see [Har77, Ex. III.10.2]). This completes the proof of the theorem.  $\square$

## 4 Generalisation of the example of Mumford

We first show how to go from a curve in  $\mathbb{P}^3$  to a curve satisfying the conditions in Theorem 3.3. This in turn gives us a clue to produce examples of non-reduced components of Hilbert scheme parametrizing smooth curves. Using a result from the previous chapter, we give several examples. In particular, we prove Theorem 1.2.

**Theorem 4.1.** Let  $d \geq 5$  be an integer,  $X$  a smooth degree  $d$  surface and  $\gamma \in H^{1,1}(X, \mathbb{Z})$ . Suppose that  $\gamma$  is the class of a (not necessarily effective) divisor  $C$  of the form  $\sum_i a_i C_i$ . Assume that  $\overline{\text{NL}}(\gamma)$  is an irreducible generically non-reduced component of  $\text{NL}_d$ . Then, for  $m \gg 0$ , there exists a smooth curve  $C'$  in the linear system corresponding to  $\mathcal{O}_X(C)(m)$  satisfying: If  $P'$  is the Hilbert polynomial of  $C'$ , there exists an irreducible generically non-reduced component of the Hilbert scheme  $H_{P'}$  containing  $C'$  such that a generic curve on this component is smooth and not contained in a surface of degree less than  $d$ .

*Proof.* Using Serre's vanishing theorem we have  $H^i(\mathcal{O}_X(C)(m - 1 - i)) = 0$  for  $m \gg 0$  and  $i \geq 1$ . Hence,  $\mathcal{O}_X(C)(m - 1)$  is globally generated. Then, [Har77, Ex. II. 7.5(d)] states that  $\mathcal{O}_X(C)(m)$  is very ample. Bertini's theorem implies that a general curve  $C'$  in the linear system corresponding to  $\mathcal{O}_X(C)(m)$  is smooth, semi-regular for  $m \gg 0$ .

A lemma of Enriques-Severi-Zariski [Har77, Corollary III.7.7], tells us that for  $m \gg d$ , we

have  $H^1(\mathcal{O}_X(-C')(d)) = H^1(\mathcal{O}_X(-C)(d-m)) = 0$ . Furthermore,  $\deg(\mathcal{O}_X(-C)(d-m)) < 0$  for  $m \gg 0$  implying that for such values of  $m$ ,  $H^0(\mathcal{O}_X(-C')(d)) = H^0(\mathcal{O}_X(-C)(d-m)) = 0$ . Denote by  $P'$  the Hilbert polynomial of  $C'$ . Then, Theorem 3.3 implies that there exists an irreducible generically non-reduced component, say  $L'$  of  $H_{P'}$  containing  $C'$  and parametrizing smooth curves.

It remains to prove that for a general  $C_g \in L'$ , there does not exist a smooth surface of smaller degree containing it. This is equivalent to saying that  $H^0(\mathcal{I}_{C_g}(k)) = 0$  for all  $k < d$ . By the upper-semicontinuity theorem, it therefore suffices to show that  $H^0(\mathcal{I}_{C'}(k)) = 0$  for all  $k < d$ . Since  $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^3}(-d)$ ,  $H^0(\mathcal{I}_X(k)) = 0$  for  $k < d$ . Since  $\deg(\mathcal{O}_X(-C')(k)) = \deg(\mathcal{O}_X(-C)(-m+k)) < 0$ ,  $H^0(\mathcal{O}_X(-C')(k)) = 0$  for  $k < d$ . So, the short exact sequence,

$$0 \rightarrow \mathcal{I}_X(k) \rightarrow \mathcal{I}_{C'}(k) \rightarrow \mathcal{O}_X(-C')(k) \rightarrow 0$$

tells us  $H^0(\mathcal{I}_{C'}(k)) = 0$  for  $k < d$ . This completes the proof of the theorem.  $\square$

We now recall some examples of non-reduced components of the Noether-Lefschetz locus.

**Theorem 4.2** ([Dan14, Theorem 6.17]). Let  $d \geq 5$  and  $C$  be a divisor in a smooth degree  $d$  surface, say  $X$ , of the form  $2l_1 + l_2$ , where  $l_1, l_2$  are coplanar lines. Let  $\gamma$  be the cohomology class of  $C$  in  $H^{1,1}(X, \mathbb{Z})$ . Then,  $\overline{\text{NL}(\gamma)}$  is a generically non-reduced component of the Noether-Lefschetz locus.

**Corollary 4.3.** Let  $d \geq 5$  be an integer,  $X$  a smooth degree  $d$  surface containing two coplanar lines  $l_1, l_2$ . Let  $C$  be a divisor in  $X$  of the form  $2l_1 + l_2$  and  $C'$  be a general element in the linear system  $|C + mH_X|$  for  $m \gg 0$ . If  $P'$  is the Hilbert polynomial of  $C'$ , there exists an irreducible generically non-reduced component of the Hilbert scheme  $H_{P'}$  containing  $C'$  such that a generic curve on this component is smooth and not contained in a surface of degree less than  $d$ .

*Proof.* Let  $\gamma'$  be the cohomology class of  $C$ . Theorem 4.2 states that  $\overline{\text{NL}(\gamma')}$  is an irreducible generically non-reduced component of  $\text{NL}_d$ . Then, Theorem 4.1 implies the corollary.  $\square$

The following lemma tells us the degree and the arithmetic genus of the curve  $C'$  as in Corollary 4.3.

**Lemma 4.4.** Let  $C'$  be as in Corollary 4.3. Then,  $\deg(C') = md + 3$  and the arithmetic genus,  $\rho_a(C') = 1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)$ .

*Proof.* Clearly,  $\deg(C') = md + 3$ . We prove the formula for the arithmetic genus. Using the adjunction formula,

$$\begin{aligned}
\rho_a(C') &= 1 + (1/2)(C'^2 + (d - 4)\deg(C')) \\
&= 1 + (1/2)(C^2 + m^2d + 2m\deg(C) + (d - 4)(md + 3)) \\
&= 1 + (1/2)(4l_1^2 + l_2^2 + 4 + md^2 + d(m^2 - 4m + 3) - 12 + 6m) \\
&= 1 + (1/2)(4(2 - d) + (2 - d) + md^2 + d(m^2 - 4m + 3) - 8 + 6m) \\
&= 1 + (1/2)(md^2 + d(m^2 - 4m - 2) + 6m + 2)
\end{aligned}$$

This proves the lemma. □

## 5 Additional remarks

**Remark 5.1.** Like in many cases, the  $m$  specified in Corollary 4.3 can be easily computed. The proof of Theorem 4.1 and hence Corollary 4.3 suggest that we simply need  $C'$  such that  $H^0(\mathcal{O}_X(-C')(d)) = 0 = H^1(\mathcal{O}_X(-C')(d))$  and  $H^1(\mathcal{O}_X(C')) = 0$ , which are the main conditions used in Theorem 3.3. We write this in the following corollary:

**Corollary 5.2.** For any  $m \geq 2d - 3$  the conclusion of Corollary 4.3 holds true.

*Proof.* Note that it suffices to find the Castelnuovo-Mumford regularity of  $\mathcal{O}_X(C)$ . Indeed, if it is equal to  $t$  then simply take  $m \geq t + 4$  and see that  $H^1(\mathcal{O}_X(C)(m)) = 0$ ,

$$0 = H^1(\mathcal{O}_X(C)(t))^\vee = H^1(\mathcal{O}_X(C)(m - 4))^\vee \stackrel{\text{SD}}{=} H^1(\mathcal{O}_X(-C)(d - m)) = H^1(\mathcal{O}_X(-C')(d)).$$

$$\text{and, } 0 = H^2(\mathcal{O}_X(C)(t))^\vee = H^0(\mathcal{O}_X(-C)(d - m)) = H^0(\mathcal{O}_X(-C')(d)).$$

Consider the short exact sequence,

$$0 \rightarrow \mathcal{O}_X(l_1 + l_2) \rightarrow \mathcal{O}_X(2l_1 + l_2) \rightarrow \mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2) \rightarrow 0$$

arising by tensoring with  $\mathcal{O}_X(l_1 + l_2)$ ,

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(l_1) \rightarrow \mathcal{O}_{l_1} \otimes \mathcal{O}_X(l_1) \rightarrow 0$$

where  $l_1, l_2$  as in Corollary 4.3. The exactness after tensor product follows from that  $\mathcal{O}_X(l_1 + l_2)$  is locally free  $\mathcal{O}_X$ -module, hence flat.

We are going to compute the Castelnuovo Mumford regularity of  $\mathcal{O}_X(l_1 + l_2)$  and  $\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)$ . We have,

**Lemma 5.3.** The sheaf  $\mathcal{O}_X(l_1 + l_2)$  is  $d - 4$ -regular.

**Lemma 5.4.** The sheaf  $\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)$  is  $2d - 7$ -regular.

This would imply that  $\mathcal{O}_X(2l_1 + l_2)$  is  $t$ -regular for  $t = \max\{2d - 7, d - 4\} = 2d - 7$ , where the last equality follows from  $d \geq 5$ .  $\square$

*Proof of Lemma 5.3.* Consider the short exact sequence,

$$0 \rightarrow \mathcal{O}_X(-l_1 - l_2) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{l_1 + l_2} \rightarrow 0.$$

[Har77, Ex. III.5.5] implies that for all  $k \in \mathbb{Z}$ , the induced map  $H^0(\mathcal{O}_X(k)) \rightarrow H^0(\mathcal{O}_{l_1 + l_2}(k))$  is surjective and  $H^1(\mathcal{O}_X(k)) = 0$ . So,

$$0 = H^1(\mathcal{O}_X(-l_1 - l_2)(k)) \stackrel{\text{SD}}{=} H^1(\mathcal{O}_X(l_1 + l_2)(d - 4 - k))^\vee, \forall k \in \mathbb{Z}.$$

In other words,  $H^1(\mathcal{O}_X(l_1 + l_2)(-k + d - 4)) = 0$  for all  $k \in \mathbb{Z}$ . Now,  $H^2(\mathcal{O}_X(l_1 + l_2)(k)) \stackrel{\text{SD}}{=} H^0(\mathcal{O}_X(-l_1 - l_2)(-k + d - 4))$  is zero if the degree of  $\mathcal{O}_X(-l_1 - l_2)(-k + d - 4)$  is less than zero, which happens if  $k > d - 6$ . This proves the lemma.  $\square$

*Proof of Lemma 5.4.* Using Serre duality, we can conclude

$$H^1(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)(k))^\vee = H^0(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(-2l_1 - l_2)(-k)(-2)).$$

Now,  $\deg(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(-2l_1 - l_2)(-k)(-2)) = l_1(-2l_1 - l_2 - (k + 2)H_X) = -2(2 - d) - 1 - (k + 2) =$

$2d - 7 - k$  where the second last equality follows from  $l_1^2 = 2 - d$  which can be computed using the adjunction formula. Therefore, for  $k > 2d - 7$ ,  $H^1(\mathcal{O}_{l_1} \otimes \mathcal{O}_X(2l_1 + l_2)(k)) = 0$ . This proves the lemma.  $\square$

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